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Hyperbolic Kac Moody Algebras and Einstein Billiards

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Abstract

We identify the hyperbolic Kac Moody algebras for which there exists a Lagrangian of gravity, dilatons and p-forms which produces a billiard that can be identified with their fundamental Weyl chamber. Because of the invariance of the billiard upon toroidal dimensional reduction, the list of admissible algebras is determined by the existence of a Lagrangian in three space-time dimensions, where a systematic analysis can be carried out since only zero-forms are involved. We provide all highest dimensional parent Lagrangians with their full spectrum of p-forms and dilaton couplings. We confirm, in particular, that for the rank 10 hyperbolic algebra, $CE_{10} = A_{15}^{(2)\wedge}$, also known as the dual of $B_8^{\wedge\wedge}$, the maximally oxidized Lagrangian is 9 dimensional and involves besides gravity, 2 dilatons, a 2-form, a 1-form and a 0-form.

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1 Introduction

It has been shown recently that the dynamics of the gravitational scale factors becomes equivalent, in the vicinity of a spacelike singularity, to that of a relativistic particle moving freely on an hyperbolic billiard and bouncing on its walls [1]-[6]. A criterion for the gravitational dynamics to be chaotic is that the billiard has a finite volume. This in turn stems from the remarkable property that the billiard can be identified with the fundamental Weyl chamber of an hyperbolic Kac Moody algebra. Some of these algebras are well known: in particular, the famous hyperbolic algebras[2] E_{10}, BE_{10}, DE_{10} [8, 9, 10] are related to strings, supergravities and M-theory; the $AE_n, n < 10$ [7] emerge from pure gravity in various dimensions and more generally, the algebras that are overextensions of finite dimensional simple Lie algebras [11, 12] - also twisted overextensions [13] - are associated to gravitational models that reduce to G/H coset models upon toroidal dimensional reduction to $D = 3$. Several other hyperbolic algebras also appear in the billiard analysis of $D = 4$ and $D = 5$ spatially homogeneous cosmological models [15]. This kind of analysis has attracted a lot of interest recently in connection with U-dualities [16] and hidden symmetries of M-theory [17, 18, 19, 20, 21, 22].

The purpose of this paper is twofold: first we select all hyperbolic Kac Moody algebras for which a billiard description exists and then we explicitly construct all Lagrangians describing gravity coupled to dilatons and p-forms producing these billiards.

We are able to give exhaustive results because (i) the hyperbolic algebras are all known and classified[3] [23], and (ii) only the finite number of algebras with rank r between 3 and 10 are relevant in this context. Note that there are infinitely many hyperbolic algebras of rank two and that there exists no hyperbolic algebra of rank $r > 10$. The analysis is considerably simplified because of the invariance of the billiard under toroidal dimensional reduction to dimensions $D \geq 3$. Indeed, as explained in [12], the billiard region stays the same, but a symmetry wall in D dimensions may become an electric or magnetic p-form wall in a lower dimension. The invariance under dimensional reduction implies in particular that the selection of algebras with a billiard description can be performed by analyzing Lagrangians in $D = 3$ dimensions.

Simplifications in $D = 3$ occur because only 0-forms are present: indeed, via appropriate dualizations, all p-forms can be reduced to 0-forms. To be concrete, for the hyperbolic algebras of real rank r between 3 and 6, we first try to reproduce their Dynkin diagram with a set of r dominant walls comprising one symmetry wall $(\beta^2 - \beta^1)$ and $(r - 1)$ scalar walls. If this can be done, we still have to check that the remaining walls are subdominant i.e., that they can be written as linear combinations of the dominant ones with positive coefficients. In particular, this analysis requires that any dominant set necessarily involves one magnetic wall and $(r - 2)$ electric walls. Note that our search for gravitational Lagrangians in $D = 3$ is systematic although no symmetry is required. To deal with the hyperbolic algebras of ranks 7 to 10, it is actually not necessary to first reduce to 3 di-

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[2]$E_{10} = E_8^{\wedge\wedge}, BE_{10} = B_8^{\wedge\wedge}, DE_{10} = D_8^{\wedge\wedge}, AE_n = A_{n-2}^{\wedge\wedge}$; more generally, the names here given to the algebras are taken from [12] and [13]; in the table of the Dynkin diagrams given in [13], the name $D_{r+1}^{(2)}$ should be replaced by $D_{r+1}^{(2)\wedge}$.

[3]Note however six missing cases in [23], two with rank 3, two with rank 4 and two with rank 5; their Dynkin diagrams are displayed at the end of the paper.

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mensions: the overextensions of finite simple Lie algebras have already been associated to billiards of some Lagrangians and for the remaining four algebras, the rules we have found in the previous cases allow to straightforwardly construct the Lagrangian in the maximal oxidation dimension.

We then analyze which three-dimensional system admits parents in higher dimensions and construct the Lagrangian in the maximal oxidation dimension. In order to do so, we take an algebra in the previous list and we determine successively the maximal spacetime dimension, the dilaton number, the p-form content and the dilaton couplings:

1. One considers the Dynkin diagram of the selected algebra and looks at the length of its "A-chain"[4], starting with the symmetry root. Our analysis produces the following oxidation rule: if the A-chain has length k, the theory can be oxidized up to

   (a) $D_{max} = k + 2$, if the next connected root has a norm squared smaller than 2,

   (b) $D_{max} = k + 1$, if the next connected root has a norm squared greater than 2.

   This generalizes the oxidation rule by [24] and [25, 26, 27], obtained by group theoretical arguments applied to coset models.

2. For given space dimension $d = D - 1$ and rank r of the algebra, the number of dilatons is given by $N = r - d$ because the dominant walls are required to be r independent linear forms in the d scale factors $\{\beta^1, ..., \beta^d\}$ and the N dilatons.

3. Because it is known how the p-form walls connect to the A-chain [12], one can read on the Dynkin diagram which p-forms[5] appear in the maximal oxidation dimension.

4. The dilaton couplings of the p-forms are computed from the norms and scalar products of the walls which have to generate the Cartan matrix of the hyperbolic algebra. This means in particular that, even if the nature of the walls changes during the oxidation procedure, their norms and scalar products remain unchanged. Note also that in all dimensions $D > 3$ the subdominant conditions are always satisfied.

As a byproduct of our analysis, we note that, for each billiard identifiable as the fundamental Weyl chamber of an hyperbolic algebra, the positive linear combinations of the dominant walls representing the subdominant ones only contain integer coefficients. Hence, the dominant walls of the Lagrangian correspond to the simple roots of the hyperbolic algebra, while the subdominant ones correspond to non simple positive roots. The gravitational theory does not give all the positive roots; even the 3-dimensional scalar Lagrangians do not describe coset spaces in general. Nevertheless, the reflections relative to the simple roots generate the Weyl group of the hyperbolic algebra; this group in turn gives an access to other positive roots and suggests that a Lagrangian capable to produce these roots via billiard walls needs more exotic fields than just p-forms.

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[4]An "A-chain" of length k is a chain of k vertices with norm squared equal to 2 and simply laced.
[5]or their dual $(d - p - 1)$-forms

Our paper is organized as follows. The general framework of our analysis is outlined in the first section: the form of the searched for gravitational Lagrangians is recalled, together with the list of their walls and the metric used to build the Cartan matrix. In the next four sections, we deal with hyperbolic algebras of rank 3 to 6. First, in $D = 3$ spacetime dimensions, we compute the 3-dimensional dilaton couplings needed to reproduce the Dynkin diagram and check the status of the subdominant walls. This is how we select the admissible algebras. Next, for each of them, we determine which Lagrangian can be oxidized and we produce it in the maximal oxidation dimension. The 18 hyperbolic algebras of ranks 7 to 10 are reviewed in the last section; as explained before, they are singled out for special treatment because 14 of them are overextensions of finite dimensional simple Lie algebras and the remaining 4 are dual to the overextension $B_n^{\wedge\wedge}$ (with $n = 5, 6, 7, 8$). We explicitly write down the $D_{max} = 9$ Lagrangian system obtained previously in [14] the billiard of which is the fundamental Weyl chamber of the algebra CE_{10}. Among the four hyperbolic algebras of rank 10, CE_{10} is special because, unlike E_{10}, BE_{10} and DE_{10}, it does not stem from supergravities. Finally, we close our paper with some conclusions.

2 General framework

The billiard analysis refers to the dynamics, in the vicinity of a spacelike singularity, of a gravitational model described by the Lagrangian[6]

$$
\begin{aligned}
\mathcal{L}_D \quad = \quad & {}^{(D)}R \star \mathbb{1} - \sum_\alpha \star d\phi^\alpha \wedge d\phi^\alpha - \\
& -\frac{1}{2} \sum_p e^{2\lambda^{(p)}(\phi)} \star F^{(p+1)} \wedge F^{(p+1)}, \quad D \geq 3
\end{aligned}
\tag{2.1}
$$

where $\lambda^{(p)}(\phi) = \sum_\alpha \lambda_\alpha^{(p)} \phi^\alpha$ and $\star\mathbb{1} = \sqrt{|{}^{(D)}g|} \, dx^0 \wedge ... \wedge dx^{D-1}$. The dilatons are denoted by $\phi^\alpha, (\alpha = 1, ..., N)$; their kinetic terms are normalized with a weight 1 with respect to the Ricci scalar. The Einstein metric has Lorentz signature $(-, +, ..., +)$; its determinant is ${}^{(D)}g$. The integer $p \geq 0$ labels the various p-forms $A^{(p)}$ present in the theory, with field strengths $F^{(p+1)} = dA^{(p)}$. If there are several p-form gauge fields with the same form degree p, we will use different letters $A^{(p)}, B^{(p)}, ..$ to distinguish them.

The rules for computing the billiards have been given in details in [6, 7, 12] to which we refer the reader. We here recall the essential tools that are used throughout the paper.

2.1 The walls

The walls bounding the billiard have different origins: some arise from the Einstein-Hilbert action and involve only the scale factors β^i, $(i = 1, ..., d)$, introduced through the Iwasawa decomposition of the space metric. They are
1) the symmetry walls

$$
w_{ij}^S(\beta) = \beta^j - \beta^i, \qquad i < j,
\tag{2.2}
$$

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[6]Compared to the notations of [12], we have put a factor of 2 in the exponents of the dilaton couplings; this way, a factor $1/2$ will be removed in front of the dilatonic part of the p-form walls.

and
2) the curvature walls

$$w_{i;jk}^{G}(\beta) = 2\beta^i + \sum_{\ell \neq i,j,k} \beta^\ell, \qquad i \neq j, i \neq k, j \neq k. \tag{2.3}$$

The others come from the energy densities of the p-forms; they depend on the scale factors and the dilatons and are
3) the electric walls

$$w_{i_1...i_p}^{E(p)}(\beta, \phi) = \beta^{i_1} + ... + \beta^{i_p} + \sum_{\alpha} \lambda_{\alpha}^{(p)} \phi^\alpha \qquad i_1 < ... < i_p \tag{2.4}$$

and
4) the magnetic walls

$$w_{i_1...i_{d-p-1}}^{M(p)}(\beta, \phi) = \beta^{i_1} + ... + \beta^{i_{d-p-1}} - \sum_{\alpha} \lambda_{\alpha}^{(p)} \phi^\alpha \qquad i_1 < ... < i_{d-p-1}. \tag{2.5}$$

Notice that upon the change of ϕ^α into $-\phi^\alpha$, the electric walls of a p-form become the magnetic walls of its dual $(d - p - 1)$-form and vice versa.
The region of hyperbolic space where the particle motion takes place is defined through the inequalties $w_{ij}^{S} \geq 0$, $w_{i;jk}^{G} \geq 0$, $w_{i_1...i_p}^{E(p)} \geq 0$ and $w_{i_1...i_{d-p-1}}^{M(p)} \geq 0$; in fact, these inequalities follow from a simpler subset, namely

$$\beta^1 \leq \beta^2 ... \leq \beta^d, \quad w_{1;23}^{G} \geq 0, \quad w_{1...p}^{E} \geq 0, \quad w_{1...(d-p-1)}^{M} \geq 0, \tag{2.6}$$

which may still be redundant. The walls forming the minimal set needed to define completely the billiard are called "dominant"; the others are referred to as subdominant. More precisely, a wall is called subdominant if it can be expressed as a linear combination with positive coefficients of the dominant ones.

2.2 The metric

Given two walls $w(\beta, \phi) = w_i \beta^i + w_\alpha \phi^\alpha = w_\mu \beta^\mu$ and $w'(\beta, \phi) = w'_i \beta^i + w'_\alpha \phi^\alpha = w'_\mu \beta^\mu$ - the $\beta^\mu (\mu = 1, ..., d, 1 + d, ..., N + d)$ here denote scale factors β^i ($i = 1, ..., d$) and dilatons, $\beta^{\alpha+d} = \phi^\alpha$ - their scalar product is defined as

$$\begin{aligned} (w|w') &= G^{\mu\nu} w_\mu w'_\nu \\ &= \sum_i (w_i w'_i) - \frac{1}{d-1}(\sum_i w_i)(\sum_j w'_j) + \sum_\alpha (w_\alpha w'_\alpha). \end{aligned} \tag{2.7}$$

The metric $G^{\mu\nu}$ is the inverse of the Lorentzian metric $G_{\mu\nu}$ defining the kinetic term of the scale factors and dilatons; as shown in (2.7), it depends explicitly on the spatial dimension d. Notice that a symmetry wall has a norm squared equal to 2. Furthermore, the p-form electric wall $w_{1...p}^{E}$ is orthogonal to all symmetry walls except one, namely, $w_{p,p+1}^{S} = \beta^{p+1} - \beta^p$; the corresponding scalar product is equal to -1.

Let $\{w_B = w_B(\beta, \phi), B = 1, ..., r\}$ denote a set of dominant walls. The enclosed billiard volume is finite if the scalar products are such that the $r \times r$ matrix

$$A_{BC} = 2\frac{(w_B|w_C)}{(w_B|w_B)} \tag{2.8}$$

is the generalized Cartan matrix of an hyperbolic Kac Moody algebra of rank r.

3 Rank 3 Hyperbolic algebras

3.1 D=3

In space dimension $d = 2$, one has a single symmetry wall, namely

$$\alpha_1 = \beta^2 - \beta^1 \tag{3.9}$$

and $N = r - d = 3 - 2 = 1$ dilaton denoted as ϕ. It is obvious that only a 0-form magnetic wall can be connected to the symmetry wall, say

$$\alpha_2 = \beta^1 - \lambda\phi. \tag{3.10}$$

Let us show that the last dominant wall has to be an electric one denoted by

$$\alpha_3 = \lambda'\phi. \tag{3.11}$$

Indeed, had one taken for dominant the magnetic wall $\tilde{\alpha}_3 = \beta^1 - \lambda'\phi$ instead of (3.11), then, its corresponding electric wall, which is precisely $\alpha_3 = \lambda'\phi$, would be dominant too because of the impossibility to write it as a linear combination with positive coefficients of the other three α_1, α_2 and $\tilde{\alpha}_3$.

Using the metric (2.7) adapted to $d = 2$, we build the matrix

$$A_{ij} = 2\frac{(\alpha_i|\alpha_j)}{(\alpha_i|\alpha_i)} \tag{3.12}$$

and obtain

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -\frac{2}{\lambda^2} & 2 & -2\frac{\lambda'}{\lambda} \\ 0 & -2\frac{\lambda}{\lambda'} & 2 \end{pmatrix} \tag{3.13}$$

which has to be identified with the generalized Cartan matrix of an hyperbolic Kac Moody algebra of rank 3. Because ϕ can be changed into $-\phi$, λ and λ' can be chosen positive. Since in such a matrix i) the non zero off-diagonal entries are negative integers and ii) not any finite or affine Lie algebra of rank 2 has an off-diagonal negative integer < -4, one immediately infers from the expression of A_{21} in (3.13) that the allowed values for λ are

$$\lambda \in \{\sqrt{2}, \, 1, \, \sqrt{2/3}, \, 1/2\}. \tag{3.14}$$

Being a symmetry wall, α_1 has norm squared equal to 2; α_2 has norm squared $\lambda^2 \leq 2$, so that, if the Dynkin diagram has an arrow between α_1 and α_2, this arrow must be directed towards α_2. Once the value of λ has been fixed, one needs to find λ' such that both $2\lambda'/\lambda$ and $2\lambda/\lambda'$ are positive integers: this leaves $\lambda = \lambda'/2, \lambda', 2\lambda'$. These values are further constrained by the condition that the subdominant walls $\tilde{\alpha}_2 = \lambda\phi$ and $\tilde{\alpha}_3 = \beta^1 - \lambda'\phi$, stay really behind the others i.e., that there exist $k > 0$ and $\ell \geq 0$ such that

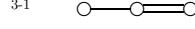$$\tilde{\alpha}_2 = k\alpha_3 \Longrightarrow \lambda/\lambda' = k \tag{3.15}$$
$$\tilde{\alpha}_3 = \alpha_2 + \ell\alpha_3 \Longrightarrow \lambda/\lambda' = \ell + 1 \tag{3.16}$$

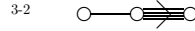which implies $k = \ell + 1 \geq 1$. Hence, the subdominant conditions require

$$\lambda' = \lambda \quad \text{or} \quad \lambda' = \lambda/2. \tag{3.17}$$

Let us summarize the 8 different possibilities for the pairs (λ, λ') that lead to Cartan matrices and draw the corresponding Dynkin diagrams:
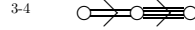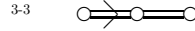
(i) for $\lambda = \sqrt{2}$ and $\lambda' = \sqrt{2}$, the Dynkin diagram describes the overextension $A_1^{\wedge\wedge}$
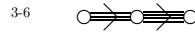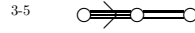
and for $\lambda = \sqrt{2}$ and $\lambda' = 1/\sqrt{2}$, the Dynkin diagram corresponds to the twisted overextension [13] $A_2^{(2)\wedge}$

(ii) $\lambda = 1$; the two possibilities are $\lambda' = 1$ and $\lambda' = 1/2$. The Dynkin diagrams are respectively

(iii) $\lambda = \sqrt{2/3}$; the two possibilities are $\lambda' = \sqrt{2/3}$ and $\lambda' = 1/\sqrt{6}$ with Dynkin diagrams given by,

(iv) $\lambda = 1/2$; the two possibilities are $\lambda' = 1/2$ and $\lambda' = 1/4$. The Dynkin diagrams are respectively,

Comments

1) When $\lambda' = \lambda$, α_2 and α_3 have to be assigned to a single scalar field; when $\lambda' \neq \lambda$, two scalars are needed in the 3-dimensional Lagrangian.
2) The algebra $(3-8)$ is missing in table 2 of reference [23]. The subalgebra obtained when removing the first or the last root is the affine $A_2^{(2)}$; the removal of the middle root gives $A_1 \times A_1$ so that this algebra satisfies indeed the criterion of hyperbolicity.
3) Remark that none of the 8 algebras above is strictly hyperbolic [7]. The latter are listed in table 1 of [23].

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[7]A strictly hyperbolic algebra is such that upon removal of a simple root, only finite Lie algebras are left behind.

3.2 D=4

The 4-dimensional Lagrangian will have no dilaton in it since $N = r - d = 0$; hence, if such a Lagrangian exists, it cannot stem from an higher dimensional parent and $D_{max} = 4$. When looking at the algebras of rank 3 selected above, one sees that only $(3 - 1)$ and $(3 - 2)$ have an A-chain of length $k = 2$ and allow, a priori, a second symmetry wall. One starts with

$$\alpha_1 = \beta^3 - \beta^2 \quad \text{and} \quad \alpha_2 = \beta^2 - \beta^1. \tag{3.18}$$

The third root may only contain β^1 and can be associated to

1. the curvature wall $\alpha_3 = 2\beta^1$ in the case of 4-dimensional pure gravity. The Dynkin diagram bears number $(3 - 1)$ above and is the overextension $A_1^{\wedge\wedge}$.

2. the electric/magnetic wall of a 1-form: $\alpha_3 = \beta^1$. This case leads to diagram $(3 - 2)$ which belongs to the twisted overextension $A_2^{(2)\wedge}$.

One sees immediately that the regions of hyperbolic space delimited by both sets of walls coincide; the difference is entirely due to the normalization of the third wall which is thus responsible for the emergence of two distinct Cartan matrices.

4 Rank 4 Hyperbolic algebras

4.1 D=3

In order to reproduce through walls the four roots of such an algebra, besides the scale factors β^1 and β^2, one needs $N = 2$ dilatons; they will be denoted as $\phi^1 = \phi$, $\phi^2 = \varphi$. There is one symmetry wall, i.e. $\alpha_1 = \beta^2 - \beta^1$ and, a priori, two choices can be made for the next three dominant walls: either (i) one takes one magnetic wall and two electric ones or (ii) one takes one electric wall and two magnetic ones. We will start with case (i) and show later how case (ii) is eliminated on account of the subdominant conditions.

4.1.1 One magnetic wall and two electric ones

The dominant walls are thus the symmetry wall

$$\alpha_1 = \beta^2 - \beta^1, \tag{4.19}$$

the magnetic wall, written as[8]

$$\alpha_2 = \beta^1 - \lambda\phi \tag{4.20}$$

and the two electric ones

$$\alpha_3 = \lambda'\phi - \mu'\varphi \tag{4.21}$$

respectively

$$\alpha_4 = \lambda''\phi + \mu''\varphi. \tag{4.22}$$

As before, the signs have already been distributed to account for the negative signs of the off-diagonal Cartan matrix elements when allowing the parameters to be either all ≥ 0 or all ≤ 0; that they can further be chosen positive is due the possibility to change ϕ^α into $-\phi^\alpha$. The general structure of the Dynkin diagram is therefore the following

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[8]This ansatz represents no loss of generality because starting from the more general expression $\alpha_2 = \beta^1 - \lambda\phi + \mu\varphi$, one can redefine the dilatons via an orthogonal transformation - leaving the dilaton Lagrangian invariant - to get the simpler expression used above.
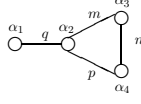
where we have not drawn the arrows and q, m, n, p are integers which count the number of lines joining two vertices.

What are the possible values that can be assigned to q, m, n and p? Since this diagram has to become the Dynkin diagram of an hyperbolic algebra, the maximal value of each of these integers is 3, because there is no finite or affine algebra of rank 3 with off-diagonal Cartan matrix elements smaller than -3. Another point is that if there were an arrow between α_1 and α_2 it necessarily points towards α_2: one has indeed $(\alpha_1, \alpha_2) = -1$, $(\alpha_1, \alpha_1) = 2$ (it is a symmetry wall), $(\alpha_2, \alpha_2) = \lambda^2$ and $A_{21} = -2/\lambda^2$ can only be $-1, -2$ or -3. We may also state that if A_{ij} is neither 0 nor -1 then $A_{ji} = -1$, because this is a common property of all finite or affine algebras of rank 3. Taking all these restrictions into account, one has to consider three different situations characterized respectively by (i) m, n, p are all different from zero, (ii) $n = 0$ and m, p are not zero (iii) $p = 0$ and n, m are not zero.

(i) If m, n and p are all non zero, then they must all be equal to 1 because, upon removal of the root α_1, one obtains a triangular diagram; now, in the set of the finite or affine algebras, there is only one such triangular Dynkin diagram and it is simply laced. That leaves, a priori, three cases labelled by the values $q = 1, 2, 3$. The corresponding dilaton couplings are

$$\lambda = \sqrt{\frac{2}{q}}; \quad \lambda' = \frac{1}{\sqrt{2q}}; \quad \mu' = \sqrt{\frac{3}{2q}}; \quad \lambda'' = \frac{1}{\sqrt{2q}}; \quad \mu'' = \sqrt{\frac{3}{2q}}. \tag{4.23}$$

The Dynkin diagrams corresponding to $q = 1$, 2 and 3 are respectively,
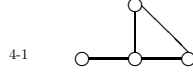
which is the overextension $A_2^{\wedge\wedge}$ and
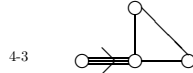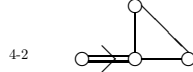
The subdominant conditions are satisfied in all cases; let us show this explicitly. With the couplings in (4.23), the dominant walls other than the symmetry wall read

$$\alpha_2 = \beta^1 - 2\frac{\phi}{\sqrt{2q}} \quad , \quad \alpha_3 = \frac{\phi}{\sqrt{2q}} - \varphi\sqrt{\frac{3}{2q}} \quad , \quad \alpha_4 = \frac{\phi}{\sqrt{2q}} + \varphi\sqrt{\frac{3}{2q}}. \tag{4.24}$$
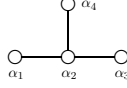
The corresponding subdominant ones are

$$\tilde{\alpha}_2 = 2\frac{\phi}{\sqrt{2q}} \quad , \quad \tilde{\alpha}_3 = \beta^1 - \frac{\phi}{\sqrt{2q}} + \varphi\sqrt{\frac{3}{2q}} \quad , \quad \tilde{\alpha}_4 = \beta^1 - \frac{\phi}{\sqrt{2q}} - \varphi\sqrt{\frac{3}{2q}} \tag{4.25}$$

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and they obey

$$\tilde{\alpha}_2 = \alpha_3 + \alpha_4 \quad , \quad \tilde{\alpha}_3 = \alpha_2 + \alpha_4 \quad , \quad \tilde{\alpha}_4 = \alpha_2 + \alpha_3. \tag{4.26}$$
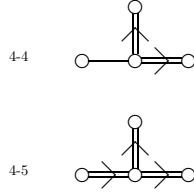
(ii) if $n = 0$ and m, p are not zero, the structure of the Dynkin diagram is the following

Comparison with the similar graphs of [23] impose (i) $m = p = 2$, (ii) $q = 1$ or $q = 2$ and (iii) an arrow pointing from α_2 to α_3 and another arrow from α_2 to α_4. Accordingly, the dilaton couplings producing them are given by

$$\lambda = \sqrt{\frac{2}{q}}; \quad \lambda' = \frac{1}{\sqrt{2q}}; \quad \mu' = \frac{1}{\sqrt{2q}}; \quad \lambda'' = \frac{1}{\sqrt{2q}}; \quad \mu'' = \frac{1}{\sqrt{2q}}. \tag{4.27}$$

The Dynkin diagrams corresponding to $q = 1$ and 2 are respectively,

Again, the subdominant conditions are fulfilled: indeed, one gets $\tilde{\alpha}_2 = \alpha_3 + \alpha_4$, $\tilde{\alpha}_3 = \alpha_2 + \alpha_4$, $\tilde{\alpha}_4 = \alpha_2 + \alpha_3$.

(iii) if $p = 0$ and n, m are not zero, the structure of the Dynkin diagram is the following
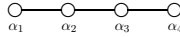
The dominant walls now simplify as

$$\alpha_1 = \beta^2 - \beta^1 \quad , \quad \alpha_2 = \beta^1 - \lambda\phi \quad , \quad \alpha_3 = \lambda'\phi - \mu'\varphi \quad , \quad \alpha_4 = \mu''\varphi. \tag{4.28}$$

We want the corresponding magnetic and electric walls to be effectively subdominant: this is indeed satisfied when 1) λ and therefore λ' are positive; 2) $\lambda'/\lambda \le 1$ (that is $\lambda'/\lambda = 1$ or $\lambda'/\lambda = 1/2$) and 3) $\mu'/\mu'' \ge \lambda'/\lambda$. Accordingly, the remaining possibilities for λ, λ' and μ' are, a priori, those given in the following table:

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	λ	λ'	μ'
1	$\sqrt{2}$	$\sqrt{2}$	$\sqrt{2}$
2.a	$\sqrt{2}$	$1/\sqrt{2}$	$\sqrt{3/2}$
2.b	$\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{2}$
2.c	$\sqrt{2}$	$1/\sqrt{2}$	$1/\sqrt{6}$
3	1	1	1
4.a	1	$1/2$	$\sqrt{3/2}$
4.b	1	$1/2$	$1/2$
4.c	1	$1/2$	$1/\sqrt{12}$
5	$\sqrt{2/3}$	$\sqrt{2/3}$	$\sqrt{2/3}$
6.a	$\sqrt{2/3}$	$1/\sqrt{6}$	$1/\sqrt{2}$
6.b	$\sqrt{2/3}$	$1/\sqrt{6}$	$1/\sqrt{6}$
6.c	$\sqrt{2/3}$	$1/\sqrt{6}$	$1/\sqrt{18}$

The different values for μ' correspond to distinct admissible values for A_{32}. Finally, for the values of μ'', we again meet two cases depending on which of A_{34} or A_{43} is equal to -1. In each case, one has still to check the subdominant conditions.

1. The two possibilities lead to a Cartan matrix: either $\mu'' = 2\sqrt{2}$ or $\mu'' = \sqrt{2}$. The former case is ruled out because the subdominant conditions cannot be satisfied. The Dynkin diagram of the remaining case describes the twisted overextension $D_3^{(2)\wedge}$
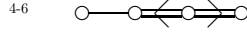
2. a. Either $\mu'' = \sqrt{2/3}$ or $\mu'' = \sqrt{6}$; both lead to hyperbolic algebras which correspond respectively to the overextension $G_2^{\wedge\wedge}$
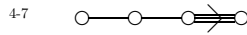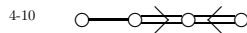
and to the twisted overextension $D_4^{(3)\wedge}$
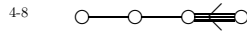
2. b. Either $\mu'' = \sqrt{1/2}$ or $\mu'' = \sqrt{2}$; the Dynkin diagrams correspond respectively to the twisted overextension $A_4^{(2)\wedge}$
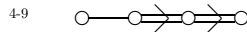
and to the overextension $C_2^{\wedge\wedge}$

10

2. c. Only the value $\mu'' = 2/\sqrt{6}$ is compatible with the subdominant conditions. The corresponding algebra is given by

4-11

3. Here again, only the value $\mu'' = 1$ can be retained on account of the subdominant conditions. This leads to

4-12

4. a. Either $\mu'' = \sqrt{3}$ or $\mu'' = 1/\sqrt{3}$; both values are admissible. They lead to

4-13

4-14

4. b. Either $\mu'' = 1/2$ or $\mu'' = 1$; both values are allowed and they give respectively

4-15

4-16

4. c. Does not correspond to any hyperbolic algebra.

5. Only the value $\mu'' = \sqrt{2/3}$ is compatible with the subdominant conditions but again there is no corresponding hyperbolic algebra.

6. a. Only the first of the 2 values $\mu'' = \sqrt{2}$ and $\mu'' = \sqrt{2}/3$ leads to an hyperbolic algebra, which is

4-17

6. b. and 6. c. do not give an hyperbolic algebra.

11

4.1.2 One electric wall and two magnetic ones

This case can be eliminated on account of the subdominant conditions. Indeed, without loss of generality, one may choose the parametrization of the dominant walls such that the electric wall takes a simple form, i.e., such that

$$
\begin{aligned}
\alpha_1 &= \beta^2 - \beta^1 \\
\alpha_2 &= \beta^1 - \lambda\,\phi - \mu\,\varphi \\
\alpha_3 &= \beta^1 - \lambda'\,\phi + \mu'\,\varphi \\
\alpha_4 &= \lambda''\,\phi.
\end{aligned} \tag{4.29}
$$

Being assumed subdominant, the electric walls associated to α_2 and α_3, namely $\tilde{\alpha}_2 = \lambda\,\phi + \mu\,\varphi$ and $\tilde{\alpha}_3 = \lambda'\,\phi - \mu'\,\varphi$ need be proportional to α_4; this happens only when $\mu = \mu' = 0$, but then (4.29) does no longer define a rank four root system.

4.2 $D > 3$

Our aim is now to determine which of the 17 algebras selected in the previous section admit Lagrangians in higher spacetime dimensions and to provide the maximal oxidation dimension and the p-forms content with its characteristic features. By considering each Dynkin diagram and looking at the length of the A-chain starting from the symmetry root α_1, we establish the following "empirical" oxidation rule: if the A-chain has length k one can oxidize the spatial dimension up to (i) $d = k + 1$ if the norm squared of the next connected root is smaller than 2 and up (ii) to $d = k$ if the norm squared of the next connected root is greater than 2. In particular, the subdominant conditions are always satisfied. Explicitly,

1. Diagram $(4 - 1)$ is the overextension $A_2^{\wedge\wedge}$. We know from [12] that it corresponds to pure gravity in $D_{max} = 5$

2. Diagrams $(4 - 2)$ and $(4 - 3)$ have an A-chain of length 1; the 3-dimensional theory cannot be oxidized.

3. Diagram $(4 - 4)$: $D_{max} = 4$. The walls are given by

$$
\begin{aligned}
\alpha_1 &= \beta^3 - \beta^2, &\qquad \alpha_2 &= \beta^2 - \beta^1, \tag{4.30} \\
\alpha_3 &= \beta^1 - \phi/\sqrt{2}, &\qquad \alpha_4 &= \beta^1 + \phi/\sqrt{2}. \tag{4.31}
\end{aligned}
$$

The last two are the electric and magnetic dominant walls of a one-form coupled to the dilaton. One sees immediately that $\tilde{\alpha}_3 = \alpha_4$ and $\tilde{\alpha}_4 = \alpha_3$.

4. Diagram $(4 - 5)$: the 3-dimensional Lagrangian has no parent in $D > 3$.

5. Diagram $(4 - 6)$ is the twisted overextension $D_3^{(2)\wedge}$. The 3-D Lagrangian cannot be oxidized the reason being that $\|\alpha_3\|^2 > 2$.

6. Diagram $(4 - 7)$ is the overextension $G_2^{\wedge\wedge}$. We know from [12] that the theory can be oxidized up to $D_{max} = 5$ where the Lagrangian is that of the Einstein-Maxwell system.

7. Diagram $(4 - 8)$ describes $D_4^{(3)\wedge}$. The A-chain has length $k = 3$ and the next connected root is longer than $\sqrt{2}$. The maximal oxidation dimension is $D_{max} = 4$ and the dominant walls are given by

$$\alpha_1 = \beta^3 - \beta^2, \qquad \alpha_2 = \beta^2 - \beta^1, \qquad (4.32)$$
$$\alpha_3 = \beta^1 - \sqrt{3/2}\phi, \qquad \alpha_4 = \sqrt{6}\phi \qquad (4.33)$$

The root α_3 is the electric wall of a 1-form, α_4 is the electric wall of a 0-form. One easily checks that the subdominant magnetic walls satisfy

$$\tilde{\alpha}_3 = \beta^1 + \sqrt{3/2}\phi = \alpha_3 + \alpha_4 \qquad (4.34)$$
$$\tilde{\alpha}_4 = \beta^1 + \beta^2 - \sqrt{6}\phi = 2\alpha_3 + \alpha_2. \qquad (4.35)$$

8. Diagram $(4 - 9)$ represents $A_4^{(2)\wedge}$. $D_{max} = 4$. Its billiard realization requires

$$\alpha_1 = \beta^3 - \beta^2, \qquad \alpha_2 = \beta^2 - \beta^1, \qquad (4.36)$$
$$\alpha_3 = \beta^1 - \sqrt{1/2}\phi, \qquad \alpha_4 = \sqrt{1/2}\phi. \qquad (4.37)$$

The last two are again the electric walls of a 1-form and a zero-form; only the dilaton couplings differ from the previous ones. The subdominant conditions are fulfilled: indeed, one obtains $\tilde{\alpha}_3 = \alpha_3 + 2\alpha_4$ and $\tilde{\alpha}_4 = 2\alpha_3 + \alpha_4 + \alpha_2$.

9. Diagram $(4 - 10)$ is the overextension $C_2^{\wedge\wedge}$. We know from [12] that the theory can be oxidized up to $D_{max} = 4$.

10. Diagram $(4 - 11)$ has $D_{max} = 4$ and

$$\alpha_1 = \beta^3 - \beta^2, \qquad \alpha_2 = \beta^2 - \beta^1, \qquad (4.38)$$
$$\alpha_3 = \beta^1 - \sqrt{1/6}\phi, \qquad \alpha_4 = \sqrt{2/3}\phi. \qquad (4.39)$$

It has the same form content as $(4 - 8)$ and $(4 - 9)$ but the dilaton couplings are still different. Again, the subdominant conditions are fulfilled: $\tilde{\alpha}_3 = \alpha_3 + \alpha_4$ and $\tilde{\alpha}_4 = 2\alpha_3 + \alpha_2$.

11. Diagrams $(4 - 12)$ to $(4 - 17)$: their 3-D Lagrangians cannot be oxidized because there is a unique root of norm squared equal to 2.

Comments

a) The subdominant conditions are indeed always fulfilled in $D > 3$ and only positive integer coefficients enter the linear combinations.
b) In case $(4 - 4)$, α_3 and α_4 are the electric and magnetic walls of the same one-form. In the other cases, they are respectively assigned to a single one-form and a single zero-form. The root multiplicity being one, there is no room for various p-forms with identical couplings.

5 Rank 5 Hyperbolic algebras

5.1 $D = 3$

The 3-dimensional Lagrangians need $N = r - d = 3$ dilatons ($\phi^1 = \phi, \phi^2 = \varphi, \phi^3 = \psi$); there are two scale factors and one symmetry wall $\alpha_1 = \beta^2 - \beta^1$. In order to reproduce the

other four simple roots of the algebra in terms of dominant walls, one has a priori three different cases to consider: indeed, the set of dominant walls can comprise (i) one magnetic wall and three electric ones, (ii) two electric walls and two magnetic ones and (iii) one electric wall and three magnetic ones. Only the first possibility will survive because as soon as the set of dominant walls contains more than one magnetic wall, one can show that the corresponding electric walls cannot fulfill the subdominant conditions. Although the proof is a straightforward generalization of the one given in subsection (4.1.2), we will provide it at the end of this section.

5.1.1 One magnetic wall and three electric ones

As in the previous sections, we use the freedom to redefine dilatons through an orthogonal transformation and choose the parametrization of the dominant walls such that

$$
\begin{align}
\alpha_1 &= \beta^2 - \beta^1 \tag{5.40}\\
\alpha_2 &= \beta^1 - \lambda\phi \tag{5.41}\\
\alpha_3 &= \lambda'\phi - \mu'\varphi \tag{5.42}\\
\alpha_4 &= \lambda''\phi + \mu''\varphi - \nu''\psi \tag{5.43}\\
\alpha_5 &= \lambda'''\phi + \mu'''\varphi + \nu'''\psi. \tag{5.44}
\end{align}
$$

One sees immediately that the symmetry root α_1 is only linked to the magnetic root α_2 while α_2 can further be connected to one, two or three roots. According to [23], five different structures for the Dynkin diagrams can be encountered; we classify them below according to the total number of roots connected to α_2; this number is 4 in case A, 3 in cases B and C, 2 in cases D and E.
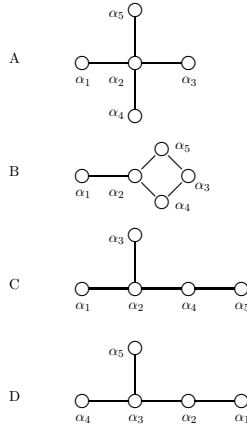
Note that C and D simply differ by the assignment of the symmetry root.
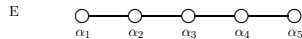
Case A - This case may be discarded. Indeed, there are in fact two hyperbolic algebras with a Dynkin diagram of that shape: one of them has a long and four short roots, while the other one has one short and four long roots. Either one cannot find couplings that

reproduce their Cartan matrix or it is the subdominant condition that is violated. More concretely:

A.1. Consider first the case for which $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ correspond to the short roots and α_5 is the long root. Then, according to [23], one needs

$$\|\alpha_1\|^2 = \|\alpha_2\|^2 = \|\alpha_3\|^2 = \|\alpha_4\|^2 = 2 \quad \text{and} \quad \|\alpha_5\|^2 = 4. \tag{5.45}$$

These conditions are immediately translated into

$$\lambda^2 = 2 \quad , \quad \lambda'^2 + \mu'^2 = 2 \quad , \quad \lambda''^2 + \mu''^2 + \nu''^2 = 2 \quad , \quad \lambda'''^2 + \mu'''^2 + \nu'''^2 = 4; \tag{5.46}$$

Hence $\lambda = \sqrt{2}$. From the shape of the diagram or equivalently from the elements of the Cartan matrix, one infers successively

1. $A_{23} = -1 = -\lambda\lambda'$ which gives $\lambda' = 1/\sqrt{2}$ and $\mu' = \sqrt{3/2}$;

2. $A_{24} = -1 = -\lambda\lambda''$ and $A_{34} = 0 = \lambda'\lambda'' - \mu'\mu''$ which gives $\lambda'' = 1/\sqrt{2}$, $\mu'' = 1/\sqrt{6}$ and $\nu'' = 2/\sqrt{3}$

3. $A_{25} = -2 = -\lambda\lambda'''$ which gives $\lambda''' = \sqrt{2}$

4. $A_{35} = 0 = \lambda'\lambda''' - \mu'\mu'''$ which gives $\mu''' = \sqrt{2/3}$ and, using the norm of α_5, $\nu''' = 2/\sqrt{3}$.

Notice that the condition $A_{45} = 0 = \lambda''\lambda''' + \mu''\mu''' - \nu''\nu'''$ is identically satisfied.
In summary, in order to fit the Dynkin diagram displayed in A (with simple lines between α_2 and $\alpha_1, \alpha_3, \alpha_4$ and a double line between α_2 and α_5 oriented towards α_2), besides the symmetry wall, we need the following set of dominant walls

$$\alpha_2 = \beta^1 - \sqrt{2}\phi \quad , \quad \alpha_4 = \frac{\phi}{\sqrt{2}} + \frac{\varphi}{\sqrt{6}} - \frac{2\psi}{\sqrt{3}} \tag{5.47}$$

$$\alpha_3 = \frac{\phi}{\sqrt{2}} - \sqrt{\frac{3}{2}}\varphi \quad , \quad \alpha_5 = \sqrt{2}\phi + \sqrt{\frac{2}{3}}\varphi + \frac{2\psi}{\sqrt{3}} \tag{5.48}$$

It is now easy to verify, for instance, that $\tilde{\alpha}_3 = \beta^1 - \frac{\phi}{\sqrt{2}} + \sqrt{\frac{3}{2}}\varphi$ cannot be written as a positive linear combination of the $\alpha_i, i = 2, ..., 5$. Accordingly, on account of the subdominant conditions, this case has to be rejected.

A.2. There is another possibility producing the same diagram as in A.1. above where the symmetry wall α_1 now plays the rôle of the long root: their norms are

$$\|\alpha_1\|^2 = 2 \quad \text{and} \quad \|\alpha_2\|^2 = \|\alpha_3\|^2 = \|\alpha_4\|^2 = \|\alpha_5\|^2 = 1 \tag{5.49}$$

but the equations giving the couplings analogous to eq.1. to eq.4. above have no solution.

A.3. In the third case, there a short and four long roots with norms

$$\|\alpha_1\|^2 = \|\alpha_2\|^2 = \|\alpha_3\|^2 = \|\alpha_4\|^2 = 2 \quad \text{and} \quad \|\alpha_5\|^2 = 1. \tag{5.50}$$

One can solve the equations for the couplings and write the following set of billiard walls: the symmetry wall $\alpha_1 = \beta^2 - \beta^1$ and

$$\alpha_2 = \beta^1 - \sqrt{2}\phi \quad , \quad \alpha_4 = \frac{\phi}{\sqrt{2}} + \frac{\varphi}{\sqrt{6}} - \frac{2\,\psi}{\sqrt{3}} \tag{5.51}$$

$$\alpha_3 = \frac{\phi}{\sqrt{2}} - \sqrt{\frac{3}{2}}\,\varphi \quad , \quad \alpha_5 = \frac{\phi}{\sqrt{2}} + \frac{\varphi}{\sqrt{6}} + \frac{\psi}{\sqrt{3}}. \tag{5.52}$$
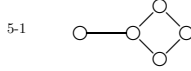
However, like in case A.1. above, one sees immediately that $\tilde{\alpha}_3 = \beta^1 - \frac{\phi}{\sqrt{2}} + \frac{3\varphi}{\sqrt{6}}$, for instance, is not subdominant; that is the reason why we discard this possibility.

Cases B - There are three hyperbolic algebras with a Dynkin diagram of this shape.
B.1. The first one admits the following couplings

$$\lambda \;=\; \sqrt{2} \;;\; \lambda' = \frac{1}{\sqrt{2}} \;;\; \mu' = \sqrt{\frac{3}{2}}$$

$$\lambda'' \;=\; 0 \quad ;\; \mu'' = \sqrt{\frac{2}{3}} \;;\; \nu'' = \frac{2}{\sqrt{3}} \;;\; \lambda''' = \frac{1}{\sqrt{2}} \;;\; \mu''' = \frac{1}{\sqrt{6}} \;;\; \nu''' = \frac{2}{\sqrt{3}} \tag{5.53}$$

and is the overextension $A_3^{\wedge\wedge}$

B.2. The second one has the following Dynkin diagram
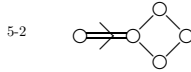
and the following set of dilaton couplings:

$$\lambda \;=\; 1 \;;\; \lambda' = \frac{1}{2} \;;\; \mu' = \frac{\sqrt{3}}{2}$$

$$\lambda'' \;=\; 0 \;;\; \mu'' = \frac{1}{\sqrt{3}} \;;\; \nu'' = \frac{\sqrt{2}}{\sqrt{3}} \;;\; \lambda''' = \frac{1}{2} \;;\; \mu''' = \frac{1}{2\sqrt{3}} \;;\; \nu''' = \frac{\sqrt{2}}{\sqrt{3}}. \tag{5.54}$$

B.3. The diagram of the third one is the same as $(5-2)$ but with the reversed arrow: this is impossible since in the present context the norms are required to satisfy $\|\alpha_1\|^2 \geq \|\alpha_2\|^2$.

Case C - In order to generate this kind of structure, one needs $\lambda''' = \mu''' = 0$ and $\lambda'\lambda'' = \mu'\mu''$. Next, from the subdominant condition for $\tilde{\alpha}_3$, we deduce that A_{32} can be -2 or -3 but since we want hyperbolic algebras, only the value $A_{32} = -2$ can be retained. Therefore $A_{23} = -1$ and $\lambda = \sqrt{2}$, $\lambda' = \lambda'' = 1/\sqrt{2}$, $\mu' = 1/\sqrt{2}$, $\mu'' = 1/\sqrt{2}$ and $\nu'' = 1$. A priori, one might still have $\nu''' = 2, 1, \sqrt{2}$ but only one value is compatible with the magnetic wall $\tilde{\alpha}_5$ being subdominant, namely $\nu''' = 1$. Accordingly, the couplings need to be defined as

$$\lambda \;=\; \sqrt{2} \;;\; \lambda' = \frac{1}{\sqrt{2}} \;;\; \mu' = \frac{1}{\sqrt{2}}$$

$$\lambda'' \;=\; \frac{1}{\sqrt{2}} \;;\; \mu'' = \frac{1}{\sqrt{2}} \;;\; \nu'' = 1 \;;\; \lambda''' = 0 \quad;\; \mu''' = 0 \quad;\; \nu''' = 1. \qquad (5.55)$$

and the Dynkin diagram is the following,
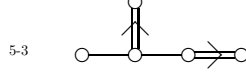
Cases D - Dynkin diagrams of this shape can only be recovered with

$$\lambda'' = \lambda''' = 0 \quad \text{and either} \quad \lambda = \sqrt{2} \quad \text{or} \quad \lambda = 1. \qquad (5.56)$$

D.1. $\lambda = \sqrt{2}$

All hyperbolic diagrams of that type have in their Cartan matrix $A_{34} = A_{43} = -1$ which means that $\|\alpha_3\|^2 = \|\alpha_4\|^2$. Two additional cases must be considered depending on which of α_2 or α_5 has a norm equal to the norm of α_3:

    1. in case D.1.1. we assume that the norms of α_3, α_4 and α_5 are equal

    2. in case D.1.2. we assume that the norms of α_2, α_3 and α_4 are equal.

The subdominant conditions here simply reduce to $A_{23} = -1$.

D.1.1. Again two hyperbolic algebras correspond to this case. For the first one, the billiard walls are built out of the following couplings

$$\lambda \;=\; \sqrt{2} \;;\; \lambda' = \frac{1}{\sqrt{2}} \;;\; \mu' = \sqrt{\frac{3}{2}}$$

$$\lambda'' \;=\; 0 \quad;\; \mu'' = \sqrt{\frac{2}{3}} \;;\; \nu'' = \frac{1}{\sqrt{3}} \;;\; \lambda''' = 0 \quad;\; \mu''' = \sqrt{\frac{2}{3}} \;;\; \nu''' = \frac{2}{\sqrt{3}} \qquad (5.57)$$

and the Dynkin diagram is that of the overextension $B_3^{\wedge\wedge}$
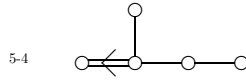
One can produce a billiard for the second one using the same couplings as in (5.57) except for

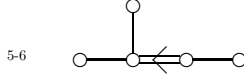$$\lambda'' \;=\; 0 \;;\; \mu'' = 2\sqrt{\frac{2}{3}} \;;\; \nu'' = \frac{2}{\sqrt{3}}. \qquad (5.58)$$

The Dynkin diagram here describes the twisted overextension $A_5^{(2)\wedge}$

5-5
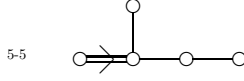
D.1.2. Here also, two hyperbolic algebras correspond to this case but one is eliminated on account of the subdominant conditions. For the remaining one, the couplings are

$$\lambda = \sqrt{2} \;;\; \lambda' = \frac{1}{\sqrt{2}} \;;\; \mu' = \frac{1}{\sqrt{2}}$$

$$\lambda'' = 0 \;;\; \mu'' = \frac{1}{\sqrt{2}} \;;\; \nu'' = \frac{1}{\sqrt{2}} \;;\; \lambda''' = 0 \;;\; \mu''' = \frac{1}{\sqrt{2}} \;;\; \nu''' = \frac{1}{\sqrt{2}} \tag{5.59}$$

and the Dynkin diagram is

5-6

D.2. $\lambda = 1$

In table 2 of reference [23], there are two hyperbolic algebras with a Dynkin diagram of this shape. Both are admissible for our present purpose:
D.2.1. The first one has couplings given by

$$\lambda = 1 \;;\; \lambda' = \frac{1}{2} \;;\; \mu' = \frac{\sqrt{3}}{2}$$

$$\lambda'' = 0 \;;\; \mu'' = \frac{2}{\sqrt{3}} \;;\; \nu'' = \sqrt{\frac{2}{3}} \;;\; \lambda''' = 0 \;;\; \mu''' = \frac{1}{\sqrt{3}} \;;\; \nu''' = \sqrt{\frac{2}{3}} \tag{5.60}$$

and corresponds to

5-7
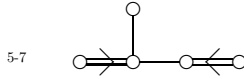
D.2.2. The second one requires

$$\lambda = 1 \;;\; \lambda' = \frac{1}{2} \;;\; \mu' = \frac{\sqrt{3}}{2}$$

$$\lambda'' = 0 \;;\; \mu'' = \frac{1}{\sqrt{3}} \;;\; \nu'' = \frac{1}{\sqrt{6}} \;;\; \lambda''' = 0 \;;\; \mu''' = \frac{1}{\sqrt{3}} \;;\; \nu''' = \sqrt{\frac{2}{3}} \tag{5.61}$$
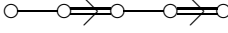
and has the following diagram

5-8
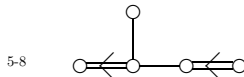
Cases E - Table 2 of reference [23] displays two hyperbolic algebras of rank 5 which are duals of each other and have linear diagrams. Only one of these two can be associated to a billiard the walls of which correspond to
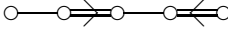E.1. $\lambda = \sqrt{2}$, $\lambda'' = \lambda''' = \mu''' = 0$ and all other dilaton couplings equal to $1/\sqrt{2}$.
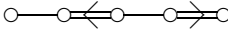Its Dynkin diagram is the twisted overextension $A_6^{(2)\wedge}$ and is given by

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There are however two more hyperbolic algebras with such linear Dynkin diagrams; they are missing in [23] but perfectly relevant in the present context:

E.2. the first one is the overextension $C_3^{\wedge\wedge}$

5-10

whose couplings are equal to the previous ones except $\nu''' = \sqrt{2}$.

E.3. The second one is the dual of $C_3^{\wedge\wedge}$ known as the twisted overextension $D_4^{(2)\wedge}$; its Dynkin diagram corresponds to the previous one with reversed arrows

5-11

and the dilaton couplings are such that $\lambda = \lambda' = \mu' = \mu'' = \nu'' = \nu''' = \sqrt{2}$ while $\lambda'' = \lambda''' = \mu''' = 0$.

5.1.2 Two or more magnetic walls

That these cases may be discarded will be proved on a particular case but the argument can be easily generalized. Suppose the dominant set comprises two magnetic walls and two electric ones, we can always choose the parametrization such that

$$
\begin{aligned}
\alpha_1 &= \beta^2 - \beta^1 \\
\alpha_2 &= \beta^1 - \lambda\,\phi - \mu\,\varphi - \nu\,\psi \\
\alpha_3 &= \beta^1 - \lambda'\,\phi - \mu'\,\varphi + \nu'\,\psi \\
\alpha_4 &= \lambda''\,\phi + \mu''\,\varphi \\
\alpha_5 &= \lambda'''\,\phi.
\end{aligned}
\tag{5.62}
$$

Being assumed subdominant, the electric walls $\tilde{\alpha}_2 = \lambda\,\phi + \mu\,\varphi + \nu\,\psi$ and $\tilde{\alpha}_3 = \lambda'\,\phi + \mu'\,\varphi - \nu'\,\psi$, independent of β^1, must be written as positive linear combinations of α_4 and α_5 only; this requires $\nu = \nu' = 0$ but then (5.62) can no longer describe a rank five root system.

The same argument remains of course valid for more magnetic walls.

5.2 $D > 3$

The empirical oxidation rule set up in the previous sections also holds for the rank 5 algebras:

1. Diagram $(5-1)$ is the Dynkin diagram of the overextension $A_3^{\wedge\wedge}$. The Lagrangian is that of pure gravity in $D_{max} = 6$.

2. Diagram $(5-2)$: the 3-dimensional Lagrangian cannot be oxidized because of the norm of α_2.

3. Diagram $(5-3)$: The Lagrangian can be oxidized twice, up to $D_{max} = 5$. The dominant walls are the three symmetry walls $\alpha_1 = \beta^4 - \beta^3$, $\alpha_2 = \beta^3 - \beta^2$, $\alpha_3 = \beta^2 - \beta^1$ and the electric wall of a 1-form

$$\alpha_4 = \beta^1 - \sqrt{1/3}\phi \qquad (5.63)$$

and its magnetic wall

$$\alpha_5 = \beta^1 + \beta^2 + \sqrt{1/3}\phi. \qquad (5.64)$$

Obviously, $\tilde{\alpha}_4 = \alpha_5$ and $\tilde{\alpha}_5 = \alpha_4$.

4. Diagram $(5-4)$ represents $B_3^{\wedge\wedge}$. The maximally oxidized Lagrangian is six-dimensional. The dominant walls are here the four symmetry walls $\alpha_1 = \beta^5 - \beta^4$, $\alpha_2 = \beta^4 - \beta^3$, $\alpha_3 = \beta^3 - \beta^2$, $\alpha_4 = \beta^2 - \beta^1$ and

$$\alpha_5 = \beta^1 + \beta^2 \qquad (5.65)$$

which is the electric or magnetic wall of a self-dual 2$-$form: obviously $\tilde{\alpha}_5 = \alpha_5$.

5. Diagram $(5-5)$ is the twisted overextension $A_5^{(2)\wedge}$. Here, $D_{max} = 4$ and besides the symmetry walls $\alpha_1 = \beta^3 - \beta^2$ and $\alpha_2 = \beta^2 - \beta^1$, one finds

$$\alpha_3 = \beta^1 - \sqrt{3/2}\phi \qquad (5.66)$$
$$\alpha_4 = \sqrt{2/3}\phi - 2/\sqrt{3}\varphi \qquad (5.67)$$
$$\alpha_5 = 2\sqrt{2/3}\varphi + 2/\sqrt{3}\psi \qquad (5.68)$$

which are the electric walls respectively of a one-form and two 0-forms. One easily checks that $\tilde{\alpha}_3 = \alpha_3 + \alpha_4 + \alpha_5$, $\tilde{\alpha}_4 = \alpha_2 + 2\alpha_3 + \alpha_5$ and $\tilde{\alpha}_5 = \alpha_2 + 2\alpha_3 + \alpha_4$.

6. Diagram $(5-6)$: $D_{max} = 4$ and one needs $\alpha_1 = \beta^3 - \beta^2$, $\alpha_2 = \beta^2 - \beta^1$ and

$$\alpha_3 = \beta^1 - \sqrt{1/2}\phi \qquad (5.69)$$
$$\alpha_4 = \sqrt{1/2}\phi - \sqrt{1/2}\varphi \qquad (5.70)$$
$$\alpha_5 = \sqrt{1/2}\phi + \sqrt{1/2}\varphi. \qquad (5.71)$$

The form-field content is the same as the previous one but the dilaton couplings are different. Moreover: $\tilde{\alpha}_3 = \alpha_3 + \alpha_4 + \alpha_5$, $\tilde{\alpha}_4 = \alpha_2 + 2\alpha_3 + \alpha_5$ and $\tilde{\alpha}_5 = \alpha_2 + 2\alpha_3 + \alpha_4$.

7. Diagrams $(5-7)$ and $(5-8)$: their 3-D Lagrangians cannot be further oxidized because of the norm of α_2.

8. Diagram $(5-9)$ describes $A_6^{(2)\wedge}$. Here, $D_{max} = 4$. One obtains the billiard with $\alpha_1 = \beta^3 - \beta^2$, $\alpha_2 = \beta^2 - \beta^1$ and

$$\alpha_3 = \beta^1 - \sqrt{1/2}\phi \qquad (5.72)$$
$$\alpha_4 = \sqrt{1/2}\phi - \sqrt{1/2}\varphi \qquad (5.73)$$
$$\alpha_5 = \sqrt{1/2}\varphi. \qquad (5.74)$$

One draws the same conclusion as for $(5-6)$ and $(5-9)$ above. Here again: $\tilde{\alpha}_3 = \alpha_3 + \alpha_4 + \alpha_5$, $\tilde{\alpha}_4 = \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5$ and $\tilde{\alpha}_5 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$.

9. Diagram $(5 - 10)$ is the overextension $C_3^{\wedge\wedge}$; the maximal oxidation dimension is $D_{max} = 4$ and the corresponding Lagrangian can be found in [12].

10. Diagram $(5 - 11)$ is the twisted overextension $D_4^{(2)\wedge}$. No Lagrangian exists in higher dimensions.

Comment

The results of this section show again that the subdominant conditions play an important rôle in three dimensions where they effectively contribute to the elimination of several Dynkin diagrams. However, once they are satisfied in three dimensions, they are always fulfilled in all dimensions where a Lagrangian exists and only integers enter the linear combinations.

6 Rank 6 Hyperbolic algebras

6.1 $D = 3$

The number of dilatons in the 3-dimensional Lagrangian is equal to $N = 4$: we denote them by $\phi^1 = \phi, \phi^2 = \varphi, \phi^3 = \psi, \phi^4 = \chi$. A straightforward generalization of the argument used in the previous sections implies that a single configuration for the set of dominant walls has to be considered. It comprises one magnetic wall and four electric ones.

After allowed simplifications, the dominant walls are parametrized according to:

$$
\begin{align}
\alpha_1 &= \beta^2 - \beta^1, \tag{6.75}\\
\alpha_2 &= \beta^1 - \lambda\phi, \tag{6.76}\\
\alpha_3 &= \lambda'\phi - \mu'\varphi, \tag{6.77}\\
\alpha_4 &= \lambda''\phi + \mu''\varphi - \nu''\psi, \tag{6.78}\\
\alpha_5 &= \lambda'''\phi + \mu'''\varphi + \nu'''\psi - \rho'''\chi, \tag{6.79}\\
\alpha_6 &= \lambda''''\phi + \mu''''\varphi + \nu''''\psi + \rho''''\chi. \tag{6.80}
\end{align}
$$

The structure of the Dynkin diagrams is therefore displayed in one of the cases labelled A to E below, depending on the number of vertices connected to α_2. When necessary, further subclasses are introduced according to the number of vertices linked to α_3,...

Case A - The central vertex is labelled α_2 and is connected to the five other vertices:
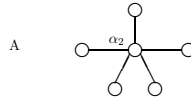
There is a single hyperbolic algebra of this type in [23]; one can solve the equations for the dilaton couplings, but the subdominant walls are not expressible as positive linear combinations of the dominant ones.
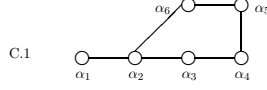
Case B - The root α_2 is connected to four vertices:

21
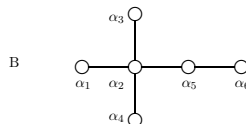
There are three hyperbolic algebras with that kind of Dynkin diagram but none of them can be retained: indeed, couplings exist but the subdominant conditions cannot be fulfilled.

**Cases C** - α_2 has three links.

One has first the loop diagram

One hyperbolic algebra has such a Dynkin diagram, namely, the overextension $A_4^{\wedge\wedge}$. As we already know from [6], the searched for 3-dimensional Lagrangian coincides with the toroidal dimensional reduction of the seven-dimensional Einstein-Hilbert Lagrangian. The dilaton couplings are given by

$$\lambda = \sqrt{2} \; ; \; \lambda' = \frac{1}{\sqrt{2}} \; ; \; \mu' = \sqrt{\frac{3}{2}} \; ; \; \lambda'' = 0 \; ; \; \mu'' = \sqrt{\frac{2}{3}} \; ; \; \nu'' = \frac{2}{\sqrt{3}} \; ; \; \lambda''' = 0 \; ; \; \mu''' = 0;$$

$$\nu''' = \frac{\sqrt{3}}{2} \; ; \; \rho''' = \frac{\sqrt{5}}{2} \; ; \; \lambda'''' = \frac{1}{\sqrt{2}} \; ; \; \mu'''' = \frac{1}{\sqrt{6}} \; ; \; \nu'''' = \frac{1}{2\sqrt{3}} \; ; \; \rho'''' = \frac{\sqrt{5}}{2} \quad (6.81)$$

and its Dynkin diagram is
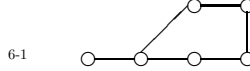
Next comes the tree diagram
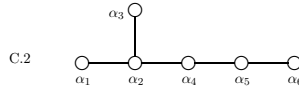
Two hyperbolic algebras have a Dynkin diagram of this shape; but they cannot be associated to billiards again because of the impossibility to satisfy the subdominant conditions.

One also has to allow a relabelling of the vertices according to
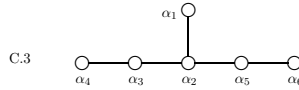
There are five hyperbolic algebras of that type; but for only one of them can one fulfill all conditions. The dilaton couplings are given by
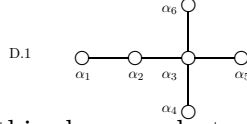
$$\lambda = \sqrt{2} \; ; \; \lambda' = \frac{1}{\sqrt{2}} \; ; \; \mu' = \sqrt{\frac{3}{2}} \; ; \; \lambda'' = 0 \; ; \; \mu'' = \sqrt{\frac{2}{3}} \; ; \; \nu'' = \frac{1}{\sqrt{3}} \; ; \; \lambda''' = \frac{1}{\sqrt{2}};$$

$$\mu''' = \frac{1}{\sqrt{6}} \; ; \; \nu''' = \frac{1}{\sqrt{3}} \; ; \; \rho''' = 1 \; ; \; \lambda'''' = 0 \; ; \; \alpha'''' = 0 \; ; \; \beta'''' = 0 \; ; \; \rho'''' = 1 \quad (6.82)$$

and its Dynkin diagram is the following

6-2
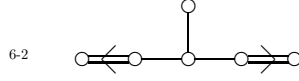
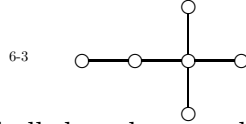**Cases D** are characterized by the fact that α_2 has two links:

D.1. corresponds further to α_3 having four links

D.1 $\quad \alpha_6 \; ; \; \alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_5 \; ; \; \alpha_4$

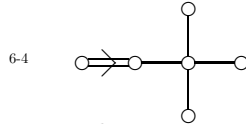Three diagrams of [23] fit in this shape; only two of them are realized through billiards. The couplings of the first one are given by

$$\lambda = \sqrt{2} \; ; \; \lambda' = \frac{1}{\sqrt{2}} \; ; \; \mu' = \sqrt{\frac{3}{2}} \; ; \; \lambda'' = 0 \; ; \; \mu'' = \sqrt{\frac{2}{3}} = \mu''' \; ; \; \nu'' = \frac{2}{\sqrt{3}}; \quad \lambda''' = 0;$$

$$\nu''' = \frac{1}{\sqrt{3}}; \quad \rho''' = 1; \quad \lambda'''' = 0; \quad \mu'''' = \sqrt{\frac{2}{3}}; \quad \nu'''' = \frac{1}{\sqrt{3}}; \quad \rho'''' = 1 \qquad (6.83)$$

they provide the Dynkin diagram which is $D_4^{\wedge\wedge}$:

6-3

For the second one, $\lambda = 1$ and all the other couplings are those given in (6.83) divided by $\sqrt{2}$. They lead to the following diagram

6-4

Case D.2. corresponds to α_3 having three connections

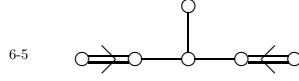D.2 $\quad \alpha_4 \; ; \; \alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_5 \;\; \alpha_6$
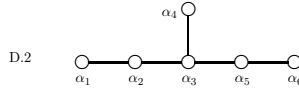
and differs from C.3. above by the assignment of the symmetry root. There are 4 Dynkin diagrams representing hyperbolic algebras of this type and they all admit a billiard.

D.2.1. The couplings are

$$\lambda = 1 \; ; \; \lambda' = \frac{1}{2} \; ; \; \mu' = \frac{\sqrt{3}}{2} \; ; \; \lambda'' = 0 \; ; \; \mu'' = \frac{1}{\sqrt{3}} \; ; \; \nu'' = \sqrt{\frac{2}{3}} \; ; \; \lambda''' = 0 \; ; \; \mu''' = \frac{1}{\sqrt{3}}$$

$$\nu''' = \frac{1}{\sqrt{6}} \; ; \; \rho''' = \frac{1}{\sqrt{2}} \; ; \; \lambda'''' = 0 \; ; \; \mu'''' = 0 \; ; \; \nu'''' = 0 \; ; \; \rho'''' = \sqrt{2}. \qquad (6.84)$$
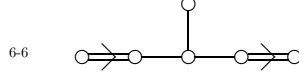
and the Dynkin diagram corresponds to

23

6-5

D.2.2. The couplings are the same as in (6.84) above except ρ'''' which reads
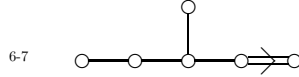
$$\rho'''' = 1/\sqrt{2}. \qquad (6.85)$$

The Dynkin diagram is

6-6

D.2.3. The dilaton couplings are given by

$$\lambda \;=\; \sqrt{2} \;;\; \lambda' = \frac{1}{\sqrt{2}} \;;\; \mu' = \sqrt{\frac{3}{2}} \;;\; \lambda'' = 0 = \;;\; \mu'' = \sqrt{\frac{2}{3}} \;;\; \nu'' = \frac{2}{\sqrt{3}} \;;\; \lambda''' = 0;$$

$$\mu''' \;=\; \sqrt{\frac{2}{3}} \;;\; \nu''' = \frac{1}{\sqrt{3}} \;;\; \rho''' = 1 \;;\; \lambda'''' = 0 \;;\; \mu'''' = 0 \;;\; \nu'''' = 0 \;;\; \rho'''' = 1 \qquad (6.86)$$

they provide the Dynkin diagram of $B_4^{\wedge\wedge}$

6-7

D.2.4. The couplings are the same as in (6.86) except

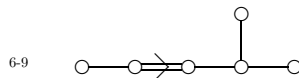$$\rho'''' = 2 \qquad (6.87)$$

and the algebra is $A_7^{(2)\wedge}$

6-8
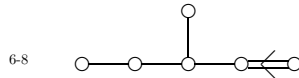
Case D.3. describes the general structure below in which α_2 and α_3 have two links while α_4 is connected three times

D.3

α_5

$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_6$
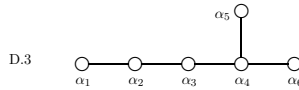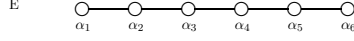
There are two hyperbolic algebras of that type but only one satisfies all billiard conditions. Its non zero couplings are

$$\lambda = \sqrt{2} \quad \text{and} \quad \lambda' = \mu' = \mu'' = \nu'' = \nu''' = \rho''' = \rho'''' = 1/\sqrt{2}. \qquad (6.88)$$
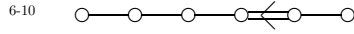
The Dynkin diagram is

6-9

24

Cases E. This set provides all linear diagrams. There are seven hyperbolic algebras of this kind and all of them are admissible

E
$\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad \alpha_4 \quad \alpha_5 \quad \alpha_6$

E.1. has the following couplings

$$\lambda = \sqrt{2} \; ; \; \lambda' = \frac{1}{\sqrt{2}} \; ; \; \mu' = \sqrt{\frac{3}{2}} \; ; \; \lambda'' = 0 \; ; \; \mu'' = \sqrt{\frac{2}{3}} \; ; \; \nu'' = \frac{2}{\sqrt{3}} \; ; \; \lambda''' = 0 ;$$

$$\mu''' = 0 \; ; \; \nu''' = \sqrt{3} \; ; \; \rho''' = 1 \; ; \; \lambda'''' = 0 \; ; \; \mu'''' = 0 \; ; \; \nu'''' = 0 \; ; \; \rho'''' = 2 \qquad (6.89)$$
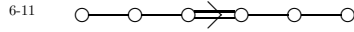
and its Dynkin diagram belongs to $E_6^{(2)\wedge}$

6-10

E.2. corresponds to

$$\lambda = \sqrt{2} \; ; \; \lambda' = \frac{1}{\sqrt{2}} \; ; \; \mu' = \sqrt{\frac{3}{2}} \; ; \; \lambda'' = 0 = \; ; \; \mu'' = \sqrt{\frac{2}{3}} \; ; \; \nu'' = \frac{1}{\sqrt{3}} \; ; \; \lambda''' = 0$$

$$\mu''' = 0 \; ; \; \nu''' = \frac{\sqrt{3}}{2} \; ; \; \rho''' = \frac{1}{2} \; ; \; \lambda'''' = 0 \; ; \; \mu'''' = 0 \; ; \; \nu'''' = 0 \; ; \; \rho'''' = 1 \qquad (6.90)$$
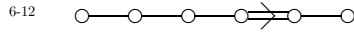
and its algebra is associated to

6-11

E.3. The walls are defined through the following set of parameters

$$\lambda = \sqrt{2} \; ; \; \lambda' = \frac{1}{\sqrt{2}} \; ; \; \mu' = \sqrt{\frac{3}{2}} \; ; \; \lambda'' = 0 \quad \; ; \; \mu'' = \sqrt{\frac{2}{3}} \; ; \; \nu'' = \frac{2}{\sqrt{3}} \; ; \; \lambda''' = 0$$

$$\mu''' = 0 \; ; \; \nu''' = \frac{\sqrt{3}}{2} \; ; \; \rho''' = \frac{1}{2} \; ; \; \lambda'''' = 0 \; ; \; \mu'''' = 0 \; ; \; \nu'''' = 0 \; ; \; \rho'''' = 1 \qquad (6.91)$$
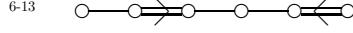
the algebra is $F_4^{\wedge\wedge}$

6-12

E.4. has the following couplings

$$\lambda = \sqrt{2} \; ; \; \lambda' = \frac{1}{\sqrt{2}} \; ; \; \mu' = \frac{1}{\sqrt{2}} \; ; \; \lambda'' = 0 = \; ; \; \mu'' = \frac{1}{\sqrt{2}} \; ; \; \nu'' = \frac{1}{\sqrt{2}} \; ; \; \lambda''' = 0$$

$$\mu''' = 0 \; ; \; \nu''' = \frac{1}{\sqrt{2}} \; ; \; \rho''' = \frac{1}{\sqrt{2}} \; ; \; \lambda'''' = 0 \; ; \; \mu'''' = 0 \; ; \; \nu'''' = 0 \; ; \; \rho'''' = \sqrt{2} \quad (6.92)$$
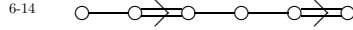
and its diagram corresponds to $C_4^{\wedge\wedge}$

25

6-13

E.5. has the same couplings as those given in (6.92) except
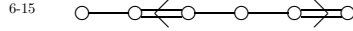
$$\rho'''' = \frac{1}{\sqrt{2}}. \tag{6.93}$$

Its diagram corresponds to $A_8^{(2)\wedge}$

6-14

E.6. is characterized by

$$\lambda = \sqrt{2} \; ; \; \lambda' = \sqrt{2} \; ; \; \mu' = \sqrt{2} \; ; \; \lambda'' = 0 \; ; \; \mu'' = \sqrt{2} \; ; \; \nu'' = \sqrt{2} \; ; \; \lambda''' = 0;$$
$$\mu''' = 0 \; ; \; \nu''' = \sqrt{2} \; ; \; \rho''' = \sqrt{2} \; ; \; \lambda'''' = 0 \; ; \; \mu'''' = 0 \; ; \; \nu'''' = 0 \; ; \; \rho'''' = \sqrt{2} \tag{6.94}$$
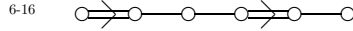
and its diagram describes $D_5^{(2)\wedge}$

6-15

E.7. is the last one of this rank; its couplings are

$$\lambda = 1 \; ; \; \lambda' = \frac{1}{2} \; ; \; \mu' = \frac{\sqrt{3}}{2} \; ; \; \lambda'' = 0 \; ; \; \mu'' = \frac{1}{\sqrt{3}} \; ; \; \nu'' = \sqrt{\frac{2}{3}} \; ; \; \lambda''' = 0;$$

$$\mu''' = 0 \; ; \; \nu''' = \frac{\sqrt{3}}{2\sqrt{2}} \; ; \; \rho''' = \frac{1}{2\sqrt{2}} \; ; \; \lambda'''' = 0 \; ; \; \mu'''' = 0 \; ; \; \nu'''' = 0 \; ; \; \rho'''' = \frac{1}{\sqrt{2}} \tag{6.95}$$

and its diagram gives $A_8^{(2)'\wedge}$

6-16

6.2 $D > 3$

Our next task is again to study which of the 16 algebras admitting a three-dimensional billiard model allow in addition a higher dimensional Lagrangian description.

1. Diagram $(6-1)$ is the overextension $A_4^{\wedge\wedge}$. The maximal oxidation dimension is $D_{max} = 7$ where the Lagrangian describes pure gravity [7].

2. Diagram $(6-2)$: Here, $D_{max} = 5$. The dominant walls are the symmetry walls $\alpha_1 = \beta^4 - \beta^3$, $\alpha_2 = \beta^3 - \beta^2$, $\alpha_3 = \beta^2 - \beta^1$ and

$$\alpha_4 = \beta^1 - 1/\sqrt{3}\phi, \tag{6.96}$$
$$\alpha_5 = \beta^1 + \beta^2 + 1/\sqrt{3}\phi - \psi, \tag{6.97}$$
$$\alpha_6 = \psi. \tag{6.98}$$

These are respectively the electric walls of a one-form, a two-form and a zero-form. One easily checks that $\tilde{\alpha}_4 = \alpha_5 + \alpha_6$, $\tilde{\alpha}_5 = \alpha_4 + \alpha_6$ and $\tilde{\alpha}_6 = \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$.

3. Diagram $(6-3)$ is the overextension $D_4^{\wedge\wedge}$; its 3-D version can be oxidized up to $D_{max} = 6$ and the Lagrangian is written in references [11] and [12].

4. Diagrams $(6-4)$, $(6-5)$ and $(6-6)$: their Lagrangians have no higher dimensional parent.

5. Diagram $(6-7)$ is the overextension $B_4^{\wedge\wedge}$. Remark that since the diagram has a fork one can oxidize in two different ways: both lead to $D_{max} = 6$. The Lagrangians can again be found in references [11] and [12].

6. Diagram $(6-8)$ is the twisted overextension $A_7^{(2)\wedge}$. $D_{max} = 6$. The dominant walls are the symmetry walls $\alpha_1 = \beta^5 - \beta^4$, $\alpha_2 = \beta^4 - \beta^3$, $\alpha_3 = \beta^3 - \beta^2$ and $\alpha_4 = \beta^2 - \beta^1$ and

$$\alpha_5 = \beta^1 + \beta^2 - \phi, \tag{6.99}$$
$$\alpha_6 = 2\phi \tag{6.100}$$

which are the electric walls of a 2-form and a 0-form. Their respective magnetic walls are subdominant: indeed one finds

$$\tilde{\alpha}_5 = \beta^1 + \beta^2 + \phi = \alpha_5 + \alpha_6 \tag{6.101}$$
$$\tilde{\alpha}_6 = \beta^1 + \beta^2 + \beta^3 + \beta^4 - 2\phi = 2\alpha_5 + \alpha_4 + 2\alpha_3 + \alpha_2. \tag{6.102}$$

7. Diagram $(6-9)$: $D_{max} = 4$. The wall system reads $\alpha_1 = \beta^3 - \beta^2, \alpha_2 = \beta^2 - \beta^1$ and

$$\alpha_3 = \beta^1 - 1/\sqrt{2}\phi, \tag{6.103}$$
$$\alpha_4 = 1/\sqrt{2}(\phi - \psi), \tag{6.104}$$
$$\alpha_5 = 1/\sqrt{2}(\psi - \chi) \tag{6.105}$$
$$\alpha_6 = 1/\sqrt{2}(\psi + \chi); \tag{6.106}$$

the last four are the electric walls of a 1-form and three 0-forms. The subdominant condition is fulfilled: indeed, one finds $\tilde{\alpha}_3 = \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$, $\tilde{\alpha}_4 = \alpha_2 + 2\alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\tilde{\alpha}_5 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_6$, $\tilde{\alpha}_6 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$.

8. Diagram $(6-10)$ is the twisted overextension $E_6^{(2)\wedge}$. The oxidation rule gives the maximal dimension $D_{max} = 5$. The walls other than the symmetry ones are

$$\alpha_4 = \beta^1 - 2/\sqrt{3}\phi, \tag{6.107}$$
$$\alpha_5 = \sqrt{3}\phi - \varphi \tag{6.108}$$
$$\alpha_6 = 2\varphi. \tag{6.109}$$

One checks that $\tilde{\alpha}_4 = \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$, $\tilde{\alpha}_5 = \alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 + \alpha_6$, $\tilde{\alpha}_6 = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5$.

9. Diagram $(6-11)$: a Lagrangian exists in $D_{max} = 5$ which produces besides the symmetry walls

$$\alpha_4 = \beta^1 - 1/\sqrt{3}\phi, \tag{6.110}$$
$$\alpha_5 = \sqrt{3}/2\phi - 1/2\varphi \tag{6.111}$$
$$\alpha_6 = \varphi. \tag{6.112}$$

The subdominant conditions read $\tilde{\alpha}_4 = \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$, $\tilde{\alpha}_5 = \alpha_2 + 2\alpha_3 + 3\alpha_4 + \alpha_5 + \alpha_6$ and $\tilde{\alpha}_6 = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5$.

10. Diagram $(6-12)$ is the overextension $F_4^{\wedge\wedge}$; the maximally oxidized theory is 6 dimensional and contains the metric, one dilaton, one zero-form, two one-forms, a two-form and a self- dual three-form field strength [11, 12].

11. Diagram $(6-13)$ is the overextension $C_4^{\wedge\wedge}$ [11]. This is the last one of its series: remember that the $C_n^{\wedge\wedge}$ algebras are hyperbolic only for $n \leq 4$. The maximal oxidation dimension is $D_{max} = 4$; besides the symmetry walls, the other dominant ones are

$$
\begin{align}
\alpha_3 &= \beta^1 - 1/\sqrt{2}\phi, \tag{6.113}\\
\alpha_4 &= 1/\sqrt{2}(\phi - \varphi), \tag{6.114}\\
\alpha_5 &= 1/\sqrt{2}(\varphi - \psi) \tag{6.115}\\
\alpha_6 &= \sqrt{2}\psi. \tag{6.116}
\end{align}
$$

The subdominant conditions are satisfied, they read $\tilde{\alpha}_3 = \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$, $\tilde{\alpha}_4 = \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + \alpha_6$, $\tilde{\alpha}_5 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$, $\tilde{\alpha}_6 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5$.

12. Diagram $(6-14)$ is the twisted overextension $A_8^{(2)\wedge}$. There is no higher dimensional theory.

13. Diagram $(6-15)$ represents $D_5^{(2)\wedge}$. In $D_{max} = 4$, the dominant walls other than the symmetry ones are given by

$$
\begin{align}
\alpha_3 &= \beta^1 - 1/\sqrt{2}\phi, \tag{6.117}\\
\alpha_4 &= 1/\sqrt{2}(\phi - \varphi), \tag{6.118}\\
\alpha_5 &= 1/\sqrt{2}(\varphi - \psi) \tag{6.119}\\
\alpha_6 &= 1/\sqrt{2}\psi. \tag{6.120}
\end{align}
$$

One obtains easily the following expressions $\tilde{\alpha}_3 = \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$, $\tilde{\alpha}_4 = \alpha_2 + 2\alpha_3 + \alpha_4 + 2\alpha_5 + 2\alpha_6$, $\tilde{\alpha}_5 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6$, $\tilde{\alpha}_6 = \alpha_2 + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6$.

14. Diagram $(6-16)$ describes $A_8^{(2)\prime\wedge}$; it cannot be associated to a billiard in $D > 3$.

Comment

Here again, in $D > 3$, the subdominant conditions are always satisfied; it is only in $D = 3$ that their rôle is crucial in the selection of the admissible algebras. Hence, they do not add any constraint in the oxidation construction.
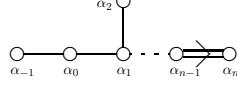
7 Rank 7, 8, 9 and 10 Hyperbolic algebras

These hyperbolic algebras fall into two classes: the first one comprises all algebras of rank $7 \leq r \leq 10$ that are overextensions of the following finite simple Lie algebras $A_n, B_n, D_n, E_6, E_7, E_8$. They are
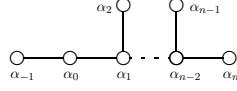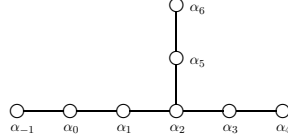
$A_n^{\wedge\wedge}, \quad (n = 5, 6, 7)$
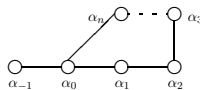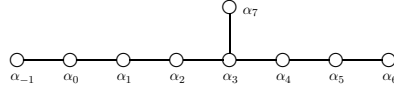
$B_n^{\wedge\wedge}, \quad (n = 5, 6, 7, 8)$

$D_n^{\wedge\wedge}, \quad (n = 5, 6, 7, 8)$
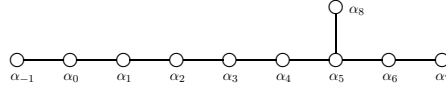

$E_6^{\wedge\wedge}$

$E_7^{\wedge\wedge}$
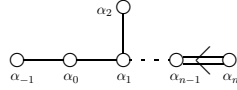

$E_8^{\wedge\wedge}$

In the second class, one finds the four duals of the $B_n^{\wedge\wedge}, (n = 5, 6, 7, 8)$, i.e. the algebras known as $CE_{n+2} = A_{2n-1}^{(2)\wedge}$

$CE_{n+2} = A_{2n-1}^{(2)\wedge}$

7.1 Overextensions of finite simple Lie algebras

The algebras of the first class have already been encountered as billiards of some three-dimensional G/H coset theories as explained in [12] to which we refer for more information. Those of rank 10, E_{10}, BE_{10} and DE_{10} have been found [8] to describe the billiards of the seven string theories, M, IIA, IIB, I, HO, HE and the closed bosonic string in 10 dimensions. More precisely, these theories split into three separate blocks which correspond to three distinct billiards: namely, $\mathcal{B}_2 = \{M, IIA, IIB\}$ leads to E_{10}, $\mathcal{B}_1 = \{I, HO, HE\}$ corresponds to BE_{10} and $\mathcal{B}_0 = \{D = 10\,\text{closed bosonic string}\}$ gives DE_{10}.

For sake of completeness, we here simply recall the maximal spacetime dimensions and the specific p-forms menus producing the billiards.

1. $A_n^{\wedge\wedge}, \quad (n = 5, 6, 7)$: the Lagrangian is that of pure gravity in $D_{max} = n + 3$.

2. $B_n^{\wedge\wedge}$, $\quad (n = 5, 6, 7, 8)$: the maximally oxidized Lagrangian lives in $D_{max} = n + 2$ where it comprises a dilaton, a 1-form coupled to the dilaton with coupling equal to $\lambda^{(1)}(\phi) = \phi/\sqrt{d-1}$ and a 2-form coupled to the dilaton with coupling equal to $\lambda^{(2)}(\phi) = 2\phi/\sqrt{d-1}$.

3. $D_n^{\wedge\wedge}$, $\quad (n = 5, 6, 7, 8)$: a Lagrangian exists in $D_{max} = n + 2$ and comprises a dilaton and a 2-form coupled to the dilaton with coupling equal to $\lambda^{(2)}(\phi) = 2\phi/\sqrt{d-1}$.

4. $E_6^{\wedge\wedge}$: the maximal oxidation dimension is $D_{max} = 8$. The Lagrangian has a dilaton, a 0-form with coupling $\lambda^{(0)}(\phi) = \phi\sqrt{2}$ and a 3-form with coupling $\lambda^{(3)}(\phi) = -\phi/\sqrt{2}$.

5. $E_7^{\wedge\wedge}$: the maximal spacetime dimension is $D_{max} = 10$. The Lagrangian describes gravity and a 4-form: it is a truncation of type IIB supergravity.

6. $E_8^{\wedge\wedge}$: $D_{max} = 11$. The Lagrangian describes gravity coupled to a 3-form; it is the bosonic sector of eleven dimensional supergravity.

7.2   The algebras $CE_{n+2} = A_{2n-1}^{(2)\wedge}$

The Weyl chamber of the algebras CE_{n+2} $(n = 5, 6, 7, 8)$, which are dual to $B_n^{\wedge\wedge}$, allows a billiard realization in maximal dimension $D_{max} = n + 1 = d + 1$. The field content of the theory is the following: there are two dilatons, ϕ and φ, a 0-form coupled to the dilatons through

$$\lambda^{(0)}(\phi) = 2\sqrt{\frac{(d-1)}{d}}\,\phi - \frac{2}{\sqrt{d}}\,\varphi, \tag{7.121}$$

a one form with dilaton couplings

$$\lambda^{(1)}(\phi) = -\sqrt{\frac{d}{(d-1)}}\,\phi \tag{7.122}$$

and a 2-form with the following couplings

$$\lambda^{(2)}(\phi) = -\frac{2}{\sqrt{d(d-1)}}\,\phi - \frac{2}{\sqrt{d}}\,\varphi. \tag{7.123}$$

In particular, the Lagrangian in $D_{max} = 9$ producing the billiard identifiable as the fundamental Weyl chamber of CE_{10} corresponds to $n = 8 = d$ and is explicitly given by [14]

$$\begin{aligned} \mathcal{L}_9 &= {}^{(9)}R \star \mathbb{1} - \star d\phi \wedge d\phi - \star d\varphi \wedge d\varphi - \frac{1}{2} e^{(2\phi\sqrt{\frac{7}{2}} - \varphi\sqrt{2})} \star F^{(1)} \wedge F^{(1)} \\ &\quad - \frac{1}{2} e^{-4\phi\sqrt{\frac{2}{7}}} \star F^{(2)} \wedge F^{(2)} - \frac{1}{2} e^{-(\phi\sqrt{\frac{2}{7}} + \varphi\sqrt{2})} \star F^{(3)} \wedge F^{(3)}. \end{aligned} \tag{7.124}$$

CE_{10} is the fourth hyperbolic algebra of rank 10; contrary to the other three cited above, which belong to the class of the overtextensions of finite simple Lie algebras, its Lagrangian (7.124) does not stem from string theories.

8 Conclusions

In this paper we have presented all Lagrangian systems in which gravity, dilatons and p-forms combine in such a way as to produce a billiard that can be identified with the Weyl chamber of a given hyperbolic Kac Moody algebra. Exhaustive results have been systematically obtained by first constructing Lagrangians in three spacetime dimensions, at least for the algebras of rank $r \leq 6$. We insist on the fact that our three-dimensional Lagrangians are not assumed to realize a coset theory. We also have solved the oxidation problem and provided the Lagrangians in the maximal spacetime dimension with their p-forms content and specific dilaton couplings. It turns out that the subdominant conditions play no rôle in the oxidation analysis. The positive integer coefficients that appear when expressing the subdominant walls in terms of the dominant ones in the maximal oxidation dimension have been systematically worked out.
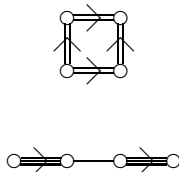
9 More hyperbolic algebras

For completeness, we draw hereafter the Dynkin diagrams of 6 hyperbolic algebras missing in reference [23]. This raises their total number to 142.
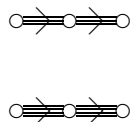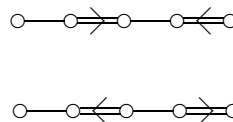
Rank 3   Rank 4   Rank 5

10 Acknowledgements

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